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DYNAMICAL POLYSYSTEMS AND OPTIMIZATION

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Dynamical Polysystems and Optimization

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D. Bushaw*

1. Introduction. The purpose of this paper is to describe the elements of a new general formalism which seems to hold promise as a device for the analysis of dynamical systems with alternative inputs, much as the now well-established subject of topological dynamics ([8]; [11], Chap. V) can be regarded as, and was originally intended to be ([3], Chap. VII) a device for the analysis of autonomous dynamical systems.

In this paper, Sections 2 and 3 present the basic definitions and a few fundamental facts; beginning in Section 4, attention is concentrated on optimization problems and the closely related question of the structure of the boundary of the so-called reachable sets. This line of inquiry culminates in the "generalized Jacobi condition" and its corollary, which appear in the last section.

The present paper touches on only a few of the potentially interesting and useful aspects of the theory of dynamical polysystems; questions of stability and "controllability", for example, are not raised at all, and must wait for discussion elsewhere.

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2. Basic Definitions. Throughout what follows, T denotes the real line with its standard topology. A quadruple (E, Φ, π, τ) will be called a dynamical polysystem (or, for short, a system) if E and Φ are topological spaces,* $\pi : \Phi \times E \times T \to E$ and $\tau : E \to T$ are continuous maps, and Axioms I-IV below are satisfied. Here and subsequently $\varphi(e, t)$ is an abbreviation for $\pi(\varphi, e, t)$, where $\varphi \in \Phi$, $e \in E$, and $t \in T$. Each φ, so regarded as a map from $E \times T$ to E, is continuous.

I. For all $\varphi \in \Phi$ and $e \in E$, $\varphi(e, 0) = e$.

II. For all $\varphi \in \Phi$, $e \in E$, and $t_1, t_2 \in T$,
$\varphi(\varphi(e, t_1), t_2) = \varphi(e, t_1 + t_2)$.

III. For all $\varphi \in \Phi$, $e \in E$, and $t \in T$, $\tau[\varphi(e,t)] = \tau(e) + t$.

IV. For all $\varphi_1, \varphi_2 \in \Phi$ and $t_o \in T$, there exists a unique $\varphi \in \Phi$, denoted by $\varphi_1/t_o/\varphi_2$, such that if $\tau(e) = t_o$, then
$\varphi(e, t) = \varphi_1(e, t)$ if $t \leqq 0$ and $= \varphi_2(e, t)$ if $t \geqq 0$.

This definition requires some comment. First, note that because of the continuity of π and because of Axioms I and II, for each fixed φ the mapping $\pi(\varphi, \cdot , \cdot) : E \times T \to E$ is a dynamical system ([11], Chap. V) or flow ([5]) on E; hence the term "polysystem". However, Axiom III along with the assumed continuity of τ prevents these separate dynamical systems from being very interesting (see Proposition 3 and the comments that follow it).

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* Throughout this paper the topology of Φ plays no role, and may therefore be taken to be discrete.

The primary intended interpretation is as follows. T is a time scale. E is the "event space" for some "plant". The typical element of E, or event, is a state-with-associated-time; the time associated with an event e is $\tau(e)$. Φ is the set of "inputs" (or "controls") φ, each of which yields a certain terminal event $\varphi(e, t)$ for any given initial event e and a time-interval of application of any given length t. The roles of the first two axioms, which are common, should be clear. Axiom III establishes a consistency between the two uses of the variable t, in timing events and measuring duration. Axiom IV, which gives a dynamical polysystem its coherence, asserts in effect that for any inputs φ_1 and φ_2 and any instant t_o, there exists an input φ which acts like φ_1 before t_o and like φ_2 afterward.

The uniqueness requirement in Axiom IV may seem severe at first, since $\varphi = \varphi_1/t_o/\varphi_2$ appears to be constrained only on the set $S_{t_o} \times T$, where $S_{t_o} = \{e \in E : \tau(e) = t_o\}$; but in fact Axiom IV leaves φ no freedom whatever. The following proposition shows this in the most important case.

Proposition 1. If $\varphi = \varphi_1/t_o/\varphi_2$ and $\tau(e) \leqq t_o \leqq \tau(e) + t$, then $\varphi(e, t) = \varphi_2[\varphi_1(e, t_o - \tau(e)), \ \tau(e) + t - t_o]$.

Proof. Let $e_1 = \varphi(e, t_o - \tau(e))$. By Axiom II,

$$\varphi(e, t) = \varphi(e_1, \tau(e) + t - t_o).$$

Since $\tau(e) + t - t_o \geqq 0$ by assumption and $\tau(e_1) = t_o$ by Axiom III, it follows that

$$\varphi(e, t) = \varphi_2(e_1, \tau(e) + t - t_o).$$

By a similar argument,

$$e_1 = \varphi_1(e, t_o - \tau(e)).$$

Elimination of e_1 from the last two equations gives the required formula.

Another by-product of Axiom IV is that it expresses the fact that a dynamical polysystem is nonanticipatory ([10], p. 6) in the following sense: if the system shifts from input φ_1 to input φ_2 at time t_o, this has no effect on events and their transformations that occur before the time t_o. In other words, phenomena in the subspace $\{e \in E : \tau(e) \leq t_o\}$ with the input $\varphi_1/t_o/\varphi_2$ do not depend on φ_2. This may be verified by the methods used to prove Proposition 1.

Two dynamical polysystems $(E_i, \Phi_i, \pi_i, \tau_i)$ $(i = 1,2)$ are <u>isomorphic</u> if there exist homeomorphisms $q : E_1 \longleftrightarrow E_2$, $r : \Phi_1 \longleftrightarrow \Phi_2$, $s : T \longleftrightarrow T$, where s is order-preserving, such that

$$\pi_2(r(\varphi_1), q(e_1), s(t)) = q[\pi_1(\varphi_1, e_1, t)]$$

for all $\varphi_1 \in \Phi_1$, $e_1 \in E_1$, and $t \in T$; and

$$\tau_2 q = s\tau_1.$$

This relation is reflexive, symmetric, and transitive, as one readily verifies.

A system is said to be a <u>state-time</u> system if $E = X \times T$, where X is some topological "state" space, and τ is the projection of E onto T. As mentioned above, this may be regarded as the archetypal situation.

<u>Proposition 2.</u> For every dynamical polysystem there exists an isomorphic state-time system.

<u>Proof</u>. Let $(E_1, \Phi_1, \pi_1, \tau_1)$ be the given system, and let $S_o = \{e \in E_1 : \tau(e) = 0\}$. Put $E_2 = S_o \times T$ and $\Phi_2 = \Phi_1$, and take r and s to be the respective identity maps. Define $q : E_1 \longleftrightarrow E_2$ by $q(e) = (\varphi_o(e, -\tau(e)), \tau(e))$, where φ_o is any fixed element of Φ_1. Then q has an inverse $\tilde{q}$ given by $\tilde{q}(e_o, t) = \varphi_o(e_o, t)$. If π_2 is defined by

$$\pi_2[\varphi, (e_o, t_o), t] = q[\pi_1(\varphi, \tilde{q}(e_o, t_o), t)] \quad (e_o \in S_o)$$

and τ_2 is taken to be the projection of E_2 onto T, then $(E_2, \Phi_2, \pi_2, \tau_2)$ is a state-time system isomorphic to $(E_1, \Phi_1, \pi_1, \tau_1)$, where the necessary homeomorphisms may be taken to be the maps q, r, s just described. The verification of the details is omitted.

Proposition 2 shows that the decision to work with "events" e in general, instead of the apparently less general state-time pairs (x, t), is not essential; it is not even a bow toward relativity theory, but only a matter of notational convenience.

Proposition 3. If (E, Φ, π, τ) is a dynamical polysystem, each of the associated dynamical systems (flows) $\varphi(\cdot\ ,\ \cdot) : E \times T \to E$ is parallelizable.

Proof. According to [5], for example, this means that for any $\varphi \in \Phi$ there exists a set $S \subset E$ and a homeomorphism h from E onto $S \times T$ such that $\varphi(S, T) = E$ and $h(\varphi(e, t)) = (e, t)$ for every $e \in S$, $t \in T$. In terms of the preceding proof, this may be accomplished by taking S to be S_o, choosing φ_o to be the given φ, and then letting h be the corresponding map q.

Thus the individual dynamical systems $\varphi(\cdot, \cdot)$ can display none of the recurrence properties with which topological dynamics is usually concerned; it is only collectively, especially as interrelated by Axiom IV, that they become qualitatively interesting.

If $t \in T$, the set $S_t = \{e \in E : \tau(e) = t\}$ may be called the t-section of E.

Proposition 4. In every dynamical polysystem, any two t-sections are homeomorphic, and every t-section is closed in E.

Proof. If t_1 and t_2 are given, then for any fixed φ, $\varphi(\cdot, t_2-t_1)$ is a homeomorphism from S_{t_1} onto S_{t_2}; its inverse is $\varphi(\cdot, t_1 - t_2)$. Here again the details may be omitted. A t-section S_t is closed because τ is continuous and $S_t = \tau^{-1}(t)$.

3. The attainable and reachable sets. Henceforth everything will refer to some fixed but otherwise unspecified dynamical polysystem (E, Φ, π, τ). If $e \in E$ and $t \in T$, the set $F(e, t) \subset E$ defined by

$$F(e, t) = \pi(\Phi, e, t) = \{\varphi(e, t) : \varphi \in \Phi\}$$

will be called the <u>attainable set form e at t</u>. The (<u>positively</u>) <u>reachable set from e</u> is the set $R(e)$ defined by

$$R(e) = \pi(\Phi, e, T^+) = \{\varphi(e, t) : \varphi \in \Phi \text{ and } t \geq 0\}.$$

(Here $T^+ = \{t \in T : t \geq 0\}$.) Since there will be no occasion to use the analogously defined negatively and bilaterally reachable sets in what follows, the adverb "positively" will not be used.

Axiomatic theories based on close counterparts of F and R have been developed by a number of writers, including Barbašin ([1]), Zubov ([16], Chap. 4), and Roxin ([14]). It is usual in these theories to make assumptions that by no means follow from the conditions stated in Section 2. For example, $F(e, t)$ is often assumed to be closed or even compact. On the other hand, some of the propositions stated below have no general analogues in the theories cited.

Clearly, $R(e) = \cup\{F(e, t) : t \in T^+\}$. A somewhat less superficial observation is:

<u>Proposition 5</u>. If $e_1 \in R(e)$, then $R(e_1) \subset R(e)$.

<u>Proof</u>. Let $e_2 \in R(e_1)$. Then there exists a $\varphi_2 \in \Phi$ and a $t_2 \in T^+$ such that $e_2 = \varphi_2(e_1, t_2)$. Likewise, there exist a $\varphi_1 \in \Phi$ and a $t_1 \in T^+$ such that $e_1 = \varphi_1(e, t_1)$. Let $\varphi = \varphi_1/(\tau(e) + t_1)/\varphi_2$. Then since

$$\tau(e) \leq \tau(e) + t_1 \leq \tau(e) + t_1 + t_2,$$

it follows from Proposition 1 that

$$\varphi(e, t_1 + t_2) = \varphi_2[\varphi_1(e, t_1), t_2] = \varphi_2(e_1, t_2) = e_2,$$

whence $e_2 \in R(e)$.

Proposition 6. For every $e \in E$ and every $t \in T$, $F(e, t)$ is pathwise connected.

Proof. The proof will be described for the case $t \in T^+$; there is an analogous proof for the complementary case. Let $\{e_1, e_2\} \subset F(e, t)$, where $t \geq 0$. Then there exist $\varphi_1, \varphi_2 \in \Phi$ such that

$$\varphi_i(e, t) = e_i \quad (i = 1, 2).$$

Define $\rho : [0, t] \to E$ by

$$\rho(t') = \varphi_2(\varphi_1(e, t-t'), t').$$

The continuity of ρ follows from that of φ_1 and φ_2. From Axiom I it follows that $\rho(0) = e_1$ and $\rho(t) = e_2$. Finally, $\rho([0, t]) \subset F(e, t)$ because for any $t' \in [0, t]$,

$$\rho(t') = \varphi_{t'}(e, t) \in F(e, t),$$

where $\varphi_{t'} = \varphi_1/(\tau(e) + t - t')/\varphi_2$. Thus $\rho([0, t])$ is a path joining e_1 and e_2 in $F(e, t)$.

It is easy to show, without using Axiom IV, that $R(e)$ is pathwise connected for every e.

The following theorem, apparently first stated (in a narrower context) by Roxin ([13], Theorem 3.1) is a strongly reminiscent of a theorem about differential equations with nonunique solutions published by Hukuhara in 1930 ([6]). I therefore call it the

Hukuhara-Roxin Theorem. If $\varphi(e, t) \in \partial R(e)$, then $\varphi(e, [0, t]) \subset \partial R(e)$. (Here ∂X denotes the boundary of X in the sense of general topology: $\partial X = \overline{X} \cap \overline{E - X}$.)

Proof. Let $\varphi(e, t) \in \partial R(e)$. Note first that $t \geq 0$; for in the contrary case, $\{e' \in E : \tau(e') < \tau(e)\}$ would be a neighborhood of $\varphi(e, t)$ disjoint from $R(e)$. Suppose that the conclusion of the theorem is false. Then there exists a $t' \in [0, t)$ such that $e' = \varphi(e, t')$ belongs to the interior of $R(e)$. By the continuity of φ in e, there exists a neighborhood N of $e_1 = \varphi(e, t)$ such that

$$\varphi(N, t' - t) \subset R(e).$$

Let $e_2 \in N$. Then

$$e_3 = \varphi(e_2, t' - t) \in R(e).$$

Thus there exists a $\psi \in \Phi$ such that for some $t_1 \geq 0$,

$$e_3 = \psi(e, t_1).$$

Note that by Axiom III, $\tau(e_3) = \tau(e_2) + t' - t = \tau(e) + t_1$. Let $\varphi_1 = \psi/\tau(e_3)/\varphi$. Since

$$\tau(e) + [\tau(e_2) - \tau(e)] = \tau(e_3) + (t - t') \geq \tau(e_3)$$

while

$$\tau(e) = \tau(e_3) - t_1 \leq \tau(e_3),$$

it follows from Proposition 1 that

$$\varphi_1(e, \tau(e_2) - \tau(e)) = \varphi(e_2, 0) = e_2.$$

Hence, since $\tau(e_2) - \tau(e) = t_1 + (t - t') \geq 0$, $e_2 \in R(e)$. Thus $N \subset R(e)$. This contradicts the assumption that $e_1 \in \partial R(e)$.

4. Optimization Problems. Within recent years rapidly rising interest has attached to problems of optimal control, where the typical objective is to choose an input (from a given set of alternatives) for a plant so that the output satisfies certain side conditions while some numerical "performance index" is maximized or minimized. (For the literature on this subject through 1961, see [7].) Optimal control problems from a rather wide class can be formulated in the context of dynamical polysystems - indeed, this is the *raison d'être* for the latter - and the principal goals of the remainder of this paper are to show how this can be done and to present some fundamental facts about the resulting situation.

One type of optimal control problem goes as follows. A dynamical polysystem, representing some plant, is given. An initial event e_o and a desired terminal event $e_1[\tau(e_1) > \tau(e_o)]$ are prescribed. Assume that

$$\Psi = \{\varphi \in \Phi : \varphi(e_o, \tau(e_1) - \tau(e_o)) = e_1\} \neq \emptyset,$$

i.e., that $e_1 \in R(e_o)$. There is given a real-valued function L on E (normally continuous and non-negative). Then

$$C(\varphi) = \int_{\tau(e_o)}^{\tau(e_1)} L(\varphi(e_o, t))dt$$

defines a real functional on Ψ. The problem is to find a φ (if it exists) that minimizes C relative to Ψ. The form of the problem can be simplified by embedding the given system in a larger one. Let $E' = E \times R$, where R is the real line, and define $\pi : E' \times \Phi \times T \to E'$ and $\tau' : E' \to T$ by

$$\pi'[\varphi, (e, r), t] = [\varphi(e, t), r + \int_{\tau(e)}^{\tau(e)+t} L(\varphi(e, t'))dt'],$$

$$\tau'(e, r) = \tau(e).$$

Then if L is well enough behaved to assure the required continuity of π', (E', Φ, π', τ') is a dynamical polysystem, and the original problem is equivalent to: Let $e_o' = (e_o, 0)$ and $G_1 = \{e_1\} \times R$, and define $J : E' \to R$ by $J(e, r) = r$; find a $\varphi \in \Phi$ and* a $t \in T^+$ such that

* In this particular problem, t must be $\tau(e_1) - \tau(e_o)$, so there is really no problem of finding it, but this formulation is used for the sake of ready generalizability.

$\varphi(e_o', t) \in G_1$ and J attains a minimum at $\varphi(e_o', t)$ relative to $G_1 \cap R'(e_o')$.

Since many important types of optimal control problems can be reduced to the same form by such devices (cf. [12], Chap. I), a definition based on this form should be useful. Accordingly, an optimization problem for the system (E, Φ, π, τ) is here defined to be a triple (e_o, G_1, J), where $e_o \in E$, G_1 is a nonempty subset of E, and J is a real-valued function on G_1. A solution of such a problem is a pair $(\bar{\varphi}, \bar{t}) \in \Phi \times T^+$ such that $\bar{\varphi}(e_o, \bar{t}) \in G_1$ and

$$J[\bar{\varphi}(e_o, \bar{t})] \leq J(e_1)$$

for all $e_1 \in G_1 \cap R(e_o)$.

A desirable further generalization would be to replace e_o by a nonempty subset G_o of E and to define an optimization problem and its solutions accordingly. The material in sections 6-8 below is applicable, with slight modifications, to this more general type of problem.

The following theorem is one form of a principle apparently first stated by Bellman (for example, see [2], p. 83).

The Principle of Optimality. If $(\bar{\varphi}, \bar{t})$ is a solution of the optimization problem (e_o, G_1, J), and $e_o' = \bar{\varphi}(e_o, t)$, where $0 \leq t \leq \bar{t}$, then $(\bar{\varphi}, \bar{t} - t)$ is a solution of the optimization problem (e_o', G_1, J).

A proof by contradiction is easily carried out with the aid of Proposition 1.

5. Nondegeneracy. The optimization problem (e_0, G_1, J) will be said to be degenerate if J attains a local minimum relative to G_1 at some point of G_1. (The meaning of the term here is similar to, but not quite the same as that given by Rozonoer in [15].) It seems that most practical problems (for example, those obtained from an integral functional as in Section 4) are nondegenerate, but the "Splitting Theorem" below makes possible the reduction, in a certain sense, of many optimization problems to nondegenerate problems. Several preliminary lemmas of an elementary topological character will be needed.

Let J be a fixed real-valued function on a topological space X. The symbol $\mu(X)$ will denote the set of points of X at which J attains a local minimum on X.

Lemma 1. If J is continuous, it is constant on any connected subset of $\mu(X)$.

There is a straightforward proof by contradiction.

Lemma 2. If J is continuous and X is locally arcwise connected, $\mu(X - \mu(X)) = \emptyset$.

Proof. Suppose that $x_0 \in \mu(X - \mu(X))$. A fortiori, $x_0 \in X - \mu(X)$. There thus exists a neighborhood N of x_0 such that for all $x \in N - \mu(X)$, $J(x) \geq J(x_0)$. It may be assumed that N is arcwise connected. Now let $x_2 \in N \cap \mu(X)$. There exists an arc P joining x_0 and x_2 in N. Suppose $J(x_2) < J(x_0)$. Since $\mu(X) \cap P \subset \mu(P)$, $x_2 \in \mu(P)$. By Lemma 1, J is constant on the component of x_2 in $\mu(P)$, which is a certain half-open or closed subarc x_1x_2 of P. Hence $J(x_1) = J(x_2) < J(x_o)$, so $x_1 \neq x_o$. Then there are poinst x' of $P - \mu(P)$ such that $J(x') < J(x_o)$. However, $P - \mu(P) \subset P - \mu(X) \subset N - \mu(X)$, so $x' \in N - \mu(X)$ while $J(x') < J(x_o)$, and this contradicts the choice of N. Hence $J(x_2) \geqq J(x_o)$; that is, $J(x) \geqq J(x_o)$ for all $x \in N \cap \mu(X)$. But then $J(x) \geqq J(x_o)$ for all $x \in N = (N \cap \mu(X)) \cup (N - \mu(X))$, so $x_o \in \mu(X)$. This again is a contradiction, and the lemma is proved.

Examples can be produced to show that if the assumption of local arcwise connectedness is dropped from Lemma 2, the statement is not generally true.

Splitting Theorem. If G_1 is locally arcwise connected and J is continuous, then any solution of the optimization problem (e_o, G_1, J) is a solution of one or the other of the problems* $P_1 = (e_o, G_1 - \mu(G_1), J)$ and $P_2 = (e_o, \mu(G_1), J)$. The problem P_1 is nondegenerate, and the problem P_2 is discrete in the sense that J is constant on components of the "target" $\mu(G_1)$, so to find a solution of P_2 it is sufficient to seek out the component K (if it exists) of $\mu(G_1)$ that intersects $R(e_o)$ for which the associated value of J is least, and then find any $(\varphi, t) \in \Phi \times T^+$ such that $\varphi(e_o, t) \in K$.

Lemmas 1 and 2 give everything needed for the proof of this theorem that does not follow from the definition of a solution and very simple considerations.

The splitting theorem, together with the previous observation about the frequency of occurrence of nondegenerate problems, makes the appearance of nondegeneracy assumptions in subsequent theorems quite unobjectionable.

* If $\mu(G_1)$ is $\emptyset$ or G_1, one or the other of the problems P_1 and P_2 evaporates and the statement is trivial.

6. The Principle of Optimal Evolution. The following necessary condition for optimality was formulated and proved by Halkin ([9]) in a more restricted context. He called it

The Principle of Optimal Evolution. If $(\bar{\varphi}, \bar{t})$ is a solution of a nondegenerate optimization problem (e_o, G_1, J), then

$$\bar{\varphi}(e_o, [0, \bar{t}]) \subset \partial R(e_o).$$

Proof. Consider $e_1 = \bar{\varphi}(e_o, \bar{t})$. If $e_1 \notin \partial R(e_o)$, $R(e_o)$ is a neighborhood of e_1, so by the assumed nondegeneracy there exists an $e_1' \in G_1 \cap R(e_o)$ such that $J(e_1') < J(e_1)$. This contradicts the assumption that $(\bar{\varphi}, \bar{t})$ is a solution of (e_o, G_1, J). Thus $e_1 \in \partial R(e_o)$. The rest follows from the Hukuhara-Roxin Theorem.

This principle locates the center of interest, as far as nondegenerate optimization problems are concerned, in the boundary of $R(e_o)$. The remainder of the paper is therefore concerned with the structure of this set.

7. Conjugate Events.

Proposition 7. For every $t \geq 0$, $\partial F(e, t) \subset \partial R(e)$, where $\partial F(e, t)$ denotes the boundary of $F(e, t)$ relative to the t-section in which it lies, namely $S_{\tau(e) + t}$.

The simple proof, which is omitted, is based on the observation that $F(e, t) = R(e) \cap S_{\tau(e) + t}$.

Thus $\cup \{\partial F(e, t) : t \geq 0\} \subset \partial R(e)$, where the two ∂'s are to be interpreted appropriately. Equality does not hold in general, so it is impossible without further assumptions to draw the highly desirable conclusion that if $(\bar{\varphi}, \bar{t})$ is a solution of a nondegenerate optimization problem (e_o, G_1, J), then $\bar{\varphi}(e_o, t) \in \partial F(e_o, t)$ for all $t \in [0, \bar{t}]$.

In order to get nearer to this conclusion, it will be assumed henceforth that E is a metric space (with metric ρ), and that the following special continuity requirement is satisfied: for every $e \in E$ there exist a neighborhood N of e and a positive real number η such that the maps $\varphi(\cdot, \cdot)$ $(\varphi \in \Phi)$ are uniformly equicontinuous on $N \times [-\eta, \eta]$.

Propositions 8, 9, and 10 are in effect lemmas.

Proposition 8. For any $e_1 \in E$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\varphi \in \Phi$ and all $e \in E$ such that $\rho(e_1, e) < \delta$,

$$\rho[e_1, \varphi(e, \tau(e_1) - \tau(e))] < \varepsilon.$$

Proof. By the special continuity assumption, there exist a neighborhood N of e_1, an $\eta > 0$, and a $\delta_1 > 0$ such that

(1) for all $\varphi \in \Phi$, $\rho[\varphi(e', t), \varphi(e, t)] < \varepsilon$ if

$$e, e' \in N; \ |t| \leqq \eta; \text{ and } \rho(e', e) < \delta_1.$$

It may be assumed that ε is so small that

(2) $B(e_1, \varepsilon) \subset N$ and $\varepsilon < \eta$;

here $B(e_1, \varepsilon)$ denotes the open ball of center e_1 and radius ε in E. Having settled (2), one may go on to assume that

(3) $\qquad \delta_1 < \varepsilon.$

By the special continuity again, there exists a $\delta_2 > 0$ such that

(4) $\qquad \delta_2 < \epsilon\, \delta_1$

and

(5) for all $\varphi \in \Phi$, $\rho(\varphi(e_1, t), e_1) < \frac{1}{2}\, \delta_1$ if $|t| < \delta_2$.

Since τ is continuous at e_1, there exists a $\delta > 0$ such that

(6) $\qquad \delta < \delta_2$

and

(7) $\qquad |\tau(e) - \tau(e_1)| < \delta_2$ if $\rho(e, e_1) < \delta$.

Thus δ has the required properties. In fact, for any $\varphi \in \Phi$ and any $e \in B(e_1, \delta)$, $|\tau(e) - \tau(e_1)| < \delta_2$ by (7). Therefore if $e_2 = \varphi(e_1, \tau(e) - \tau(e_1))$, it follows from (5) that

(8) $\qquad \rho(e_2, e_1) < \frac{1}{2}\, \delta_1.$

Since $\rho(e, e_1) < \delta < \delta_2 < \frac{1}{2}\, \delta_1$, (8) implies

(9) $\qquad \rho(e_2, e) < \delta_1.$

Also, by (8), $\rho(e, e_1) < \delta$, (3), and (2),

$$e \in N \text{ and } e_2 \in N.$$

Since $|\tau(e) - \tau(e_1)| < \delta_2 < \eta$, it now follows from (1) that

$$\rho[\varphi(e_2, \tau(e_1) - \tau(e)), \varphi(e, \tau(e_1) - \tau(e)) < \varepsilon .$$

But $\varphi(e_2, \tau(e_1) - \tau(e)) = e_1$; hence

$$\rho[e_1, \tau(e, \tau(e_1) - \tau(e))] < \varepsilon ,$$

as was to be shown.

Proposition 9. If $e_1 \in \text{int } F(e, t)$ $(t \geq 0)$, where "int" denotes the interior relative to the t-section $S_{\tau(e)+t}$ that contains $F(e, t)$, then there exists a $\delta > 0$ such that

$$\{e \in E : \rho(e, e_1) < \delta \text{ and } \tau(e) \geq \tau(e_1)\} \subset R(e).$$

Proof. Let $e_1 \in \text{int } F(e, t)$. Then there exists an $\varepsilon > 0$ such that

$$B(e_1, \varepsilon) \cap S_{\tau(e_1)} \subset F(e, t).$$

Let δ correspond to ε as in Proposition 8. Then if $e_2 \in B(e_1, \delta)$, φ is any fixed element of Φ, and $e_3 = \varphi(e_2, \tau(e_1) - \tau(e_2))$, Proposition 8 gives $e_3 \in B(e_1, \varepsilon)$. Since also $\tau(e_3) = \tau(e_1)$, $e_3 \in F(e, t)$; say $e_3 = \varphi_1(e, t)$ where $\varphi_1 \in \Phi$. Now let $\varphi_2 = \varphi_1/\tau(e_1)/\varphi$. If $\tau(e_2) \geq \tau(e_1)[\geq \tau(e)]$, Proposition 1 gives

$$\varphi_2(e, \tau(e_2)) = \varphi[\varphi_1(e, \tau(e_1) - \tau(e)), \tau(e_2) - \tau(e_1)]$$

$$= \varphi(e_3, \tau(e_2) - \tau(e_1)) = e_2.$$

Thus $e_2 \in R(e)$, as needed.

Proposition 10. If $e \in E$ and $e_1 \in \partial R(e)$, then

$$e_1 \in \overline{F(e, \tau(e_1) - \tau(e))}.$$

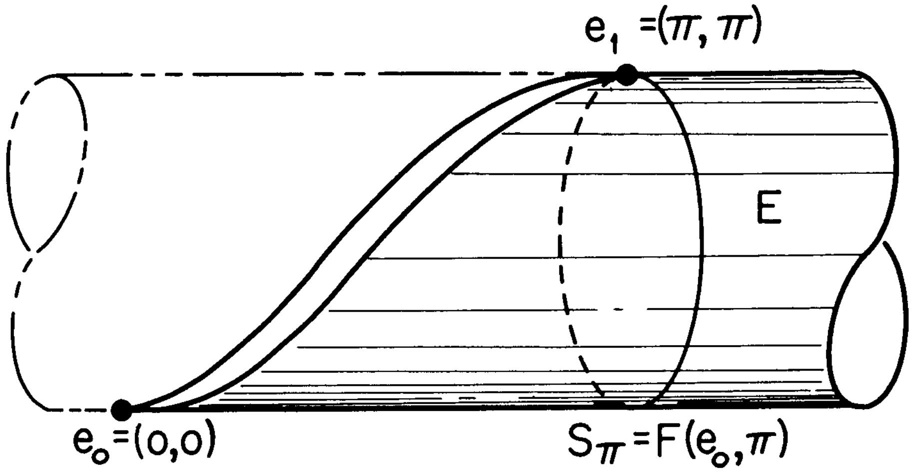
Hence $\partial R(e) \subset \cup\{\overline{F(e, t)} : t \geq 0\}$.

Proof. [The closure indicated in the proposition may be interpreted as that relative to E or that relative to $S_{\tau(e)+t}$, because this subspace is closed (Proposition 4).] Let $e \in E$ and $e_1 \in \partial R(e)$. Let $\varepsilon > 0$, and choose δ as in Proposition 8. For some $(\varphi, t) \in \Phi \times T^+$, $e_2 = \varphi(e, t) \in B(e_1, \delta)$. Now

$$e_3 = \varphi(e_2, \tau(e_1) - \tau(e_2)) = \varphi(e, \tau(e_1) - \tau(e)).$$

By Proposition 8, $e_3 \in B(e_1, \varepsilon)$. Also, $\tau(e_3) = \tau(e_1)$. Therefore $e_3 \in F(e, \tau(e_1) - \tau(e))$, so $e_1 \in \overline{F(e, \tau(e_1) - \tau(e))}$, as required.

An event e_1 will be said to be conjugate to the event e if $e_1 \in \partial R(e) \cap \text{int } F(e, \tau(e_1) - \tau(e))$, where "int" again denotes the interior relative to the proper t-section. Note that e_1 can be conjugate to e only if $\tau(e_1) > \tau(e)$.



e1 =(π, π)
E
e0=(0,0)
Sπ =F(e0,π)

Example. Consider the state-time system in which X is the unit circle with the angle $\theta \pmod{2\pi}$ as coordinate variable, Φ is the set of all functions $\varphi : T \to [-1, +1]$ that are integrable on all finite intervals,* and

$$\pi[\varphi, (\theta, t), t'] = \left(\theta + \int_{t}^{t+t'} \varphi(s)ds, \quad t + t'\right).$$

Then the event $e_1 = (\pi, \pi)$ is conjugate to the event $e = (0, 0)$ (see the figure).

The refinement of Proposition 7 toward which the preceding discussion has been directed can now be stated and quickly proved.

Boundary Decomposition Theorem. Let $C(e)$ be the set of events conjugate to e. Then $\partial R(e)$ is the union of the disjoint sets $C(e)$ and $\cup\{\partial F(e, t) : t \geq 0\}$, where the symbol $\partial F(e, t)$ is to be interpreted as in Proposition 7.

Proof. By Proposition 7,

$$\cup\{\partial F(e, t) : t \geq 0\} \subset \partial R(e),$$

and by the definition of conjugate events,

$$C(e) \subset \partial R(e).$$

Hence

$$[\cup\{\partial F(e, t) : t \geq 0\}] \cup C(e) \subset \partial R(e). \tag{10}$$

*Strictly speaking, φ's that differ on a set of measure zero should be identified.

Also by the definition of conjugate events,

$$(11) \qquad [\cup\{\partial F(e,\ t)\ :\ t \geqq 0\}] \cap C(e) = \emptyset.$$

Now suppose that $e_1 \in \partial R(e) - C(e)$. By Proposition 10, $e_1 \in \overline{F(e,\ \tau(e_1) - \tau(e))}$, where $t = \tau(e_1) - \tau(e) \geqq 0$. However, since $e_1 \notin C(e)$, $e_1 \notin \text{int } F(e,\ t)$. Thus

$$e_1 \in \overline{F(e,\ t)} - \text{int } F(e,\ t) = \partial F(e,\ t).$$

This gives

$$\partial R(e) - C(e) \subset \cup\{\partial F(e,\ t)\ :\ t \geqq 0\},$$

which, with (10) and (11), yields the desired conclusion.

8. The Generalized Jacobi Condition. The principle of optimal evolution (Section 6) of course gives a necessary condition for optimality. The concept of conjugate event makes it possible to formulate another necessary condition which, because of its formal resemblance to the classical Jacobi necessary condition of the calculus of variations (for example, see [4], p. 88), I call

The Generalized Jacobi Condition. If $(\bar{\varphi},\ \bar{t})$ is a solution of the nondegenerate optimization problem $(e_o,\ G_1,\ J)$, there are no events conjugate to e_o in the set $\bar{\varphi}(e_o,\ [0,\ \bar{t}))$.

Proof. Suppose that $(\bar{\varphi},\ \bar{t})$ is a solution such that, for some $t \in [0,\ \bar{t})$, $e_1 = \bar{\varphi}(e_o,\ t)$ is conjugate to e. Then

$$e_1 \in \operatorname{int} F(e, \tau(e_1) - \tau(e)) = \operatorname{int} F(e, t),$$

whence, by Proposition 9, there exists a $\delta > 0$ such that

$$(12) \qquad \{e' \in E : \rho(e_1, e') < \delta \text{ and } \tau(e') \geqq \tau(e)\} \subset R(e).$$

Let η satisfy $0 < \eta < \bar{t} - t$ and

$$e_2 = \bar{\varphi}(e_1, t + \eta) \in B(e_1, \delta);$$

this is possible by the continuity of φ. Now let $\delta_1 > 0$ be such that

$$(13) \qquad B(e_2, \delta_1) \subset B(e_1, \delta) \cap \{e \in E : \tau(e) > \tau(e_1)\};$$

this can be done because e_2 belongs to the set on the right, which is open. By (12) and (13),

$$B(e_2, \delta_1) \subset R(e);$$

since $e_2 \in \varphi(e, [0, t])$, this contradicts the principle of optimal evolution.

The following statement now follows from the principle of optimal evolution, the boundary decomposition theorem, and the result just proved.

Corollary. If $(\bar{\varphi}, \bar{t})$ is any solution of the nondegenerate optimization problem (e_o, G_1, J), then $\varphi(e_o, t) \in \partial F(e_o, t)$ for all $t \in [0, \bar{t})$.

This proposition is parallel to the principle of optimal evolution, but has the advantage that the necessary condition for optimality that it embodies refers only to the situation of the moment: given a knowledge of $F(e, t)$, one has to look neither forward nor backward in time in order to apply the criterion at t.

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